Comment on "PT-Symmetric versus Hermitian Formulations of Quantum Mechanics"

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Abstract

We explain why the main conclusion of Bender et al, J. Phys. A **39**, 1657 (2006), regarding the practical superiority of the non-Hermitian description of $\mathcal{PT}$-symmetric quantum systems over their Hermitian description is not valid. Recalling the essential role played by the Hermitian description in the characterization and interpretation of the physical observables, we maintain that as far as the physical aspects of the theory are concerned the Hermitian description is not only unavoidable but also indispensable.

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Recently Bender, Chen, and Milton [1] have employed the path integral method to examine the perturbative calculation of the ground-state energy and a one-point Green's function for the $\mathcal{PT}$-symmetric cubic anharmonic oscillator. They performed this calculation in both the $\mathcal{PT}$-symmetric (pseudo-Hermitian) and the Hermitian descriptions of this model and concluded that the use of the latter description leads to practical difficulties that are "severe and virtually insurmountable $\cdots$", and that such difficulties do not arise in the former description.

As explained in [2], the level of the difficulty of a calculation in $\mathcal{PT}$-symmetric quantum mechanics depends on the quantity that one chooses to calculate. If one wishes to calculate the expectation value of a canonical pair of basic observables of the theory, both the descriptions/representations involve dealing with practical problems with the same degree of difficulty. The reason why the difficulties with the $\mathcal{PT}$-symmetric representation do not surface in the interesting calculations reported in [1] is that the authors only calculate the ground-state energy and the one-point Green's function for the operator x.

The difficulties with the calculation of ground- and excited-state energies in the Hermitian representation is already clear from the complicated expression for the Hermitian Hamiltonian

that was obtained in [3, 4], and indeed the $\mathcal{PT}$-symmetric representation is much more convenient for this purpose whether one uses the path-integral or operator methods. The situation for the calculation of the expectation value of the observables such as the position which may be related to the corresponding Green's functions must be treated with more care, for the very notion of the observable for a $\mathcal{PT}$-symmetric system is a delicate issue [5, 6, 2, 3]. Specifically, for $\mathcal{PT}$-symmetric models such as the one considered in [1], the x operator is not an observable. The same holds for the unitary-equivalent operator $\tilde{x} = e^{-Q/2} x e^{Q/2}$ introduced in [1] which is manifestly non-Hermitian and consequently fails to be an observable in the Hermitian representation (conventional quantum mechanics.)

As described in [5, 7], the position observable is the nonlocal pseudo-Hermitian operator $X = e^{Q/2} x e^{-Q/2}$, and the physical one-point Green's function is $\langle 0|X|0\rangle_{\mathcal{CPT}}$. The one-point Green's function $\langle 0|x|0\rangle_{\mathcal{CPT}}$ calculated in [1], which takes an imaginary value, does not represent a physical quantity.

It is not difficult to see that the calculation of the physical one-point Green's function $\langle 0|X|0\rangle_{\mathcal{CPT}}$ is as difficult in the $\mathcal{PT}$-symmetric representation of the model as the calculation of the non-physical one-point Green's function $\langle 0|x|0\rangle_{\mathcal{CPT}}$ is in the Hermitian representation. Therefore as far as the calculation of physical quantities such as $\langle 0|X|0\rangle_{\mathcal{CPT}}$ are concerned the $\mathcal{PT}$-symmetric description is not superior to the Hermitian description.

The key ingredient that makes the Hermitian description of $\mathcal{PT}$-symmetric systems indispensable is its crucial role in determining the operators that represent the physical observables of the theory as well as its utility in providing a physical interpretation for these operators [2, 7]. In particular, once the associated Hermitian Hamiltonian is determined one can identify the underlying classical system and assign physical meaning to the Hamiltonian using its classical counterpart. This has been achieved for the $\mathcal{PT}$-symmetric cubic anharmonic oscillator in [3] and for some other toy models in [7, 8, 9, 10]. For example, it was an examination of the Hermitian representation of the $\mathcal{PT}$-symmetric cubic anharmonic oscillator that revealed the curious fact that up to cubic terms in the perturbation theory it described a Hermitian quartic anharmonic oscillator with a certain position-dependent mass [3]. See also [9, 10]. This has recently motivated a study of position-dependent mass Hermitian Hamiltonians in terms of their constant-mass non-Hermitian equivalent Hamiltonians [11]. Other evidences for the importance of the Hermitian representation is its role in the proof of the physical equivalence of the $\mathcal{PT}$-symmetric and conventional massive Thirring and Sine-Gordon field theories [12] and the recent construction of the equivalent Hermitian Hamiltonian for the $\mathcal{PT}$-symmetric Hamiltonian $H = p^2 - x^4$ defined on a complex contour [13]. The latter is particularly significant, for the equivalent Hermitian Hamiltonian turns out to be $h = 16p^2 + x^4/64 - x/2$ that is defined on $\mathbb{R}$. This observation not only explains the physical meaning of H but it also provides a direct proof of the reality, positive-definiteness, and discreteness of its spectrum, a fact whose initial proof required quite sophisticated mathematical tools [14].

In conclusion, we maintain that the importance of the Hermitian representation of the $\mathcal{PT}$-symmetric (and other) unitary quantum systems cannot be undermined by a demonstration

that certain calculations take a simpler form in the non-Hermitian representation of these systems. There are other physically relevant calculations that are at least as difficult in the non-Hermitian representation as they are in the Hermitian representation. More importantly, the quest for understanding the physical meaning and potential applications of such systems makes the Hermitian representation absolutely essential. Finally, we wish to emphasize a positive aspect of the results of [1] namely the curious identities between sums of certain Feynman diagrams that follow from the unitary-equivalence of the two representations.

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